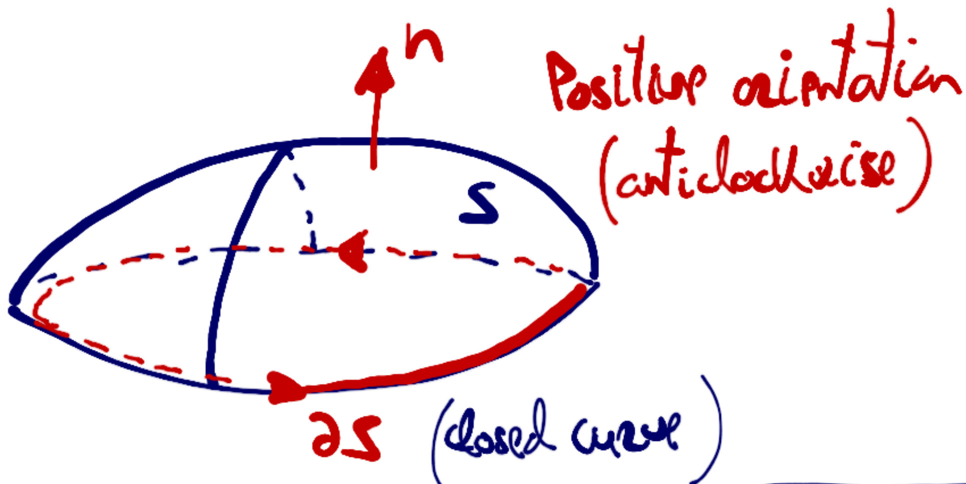
Stokes' Theorem

$S \subset \mathbb{R}^3$ surface of class C^1 (at piecewise C^1)

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ vector field. of class C^1

∂S boundary of S with compatible orientation with S.



Then,

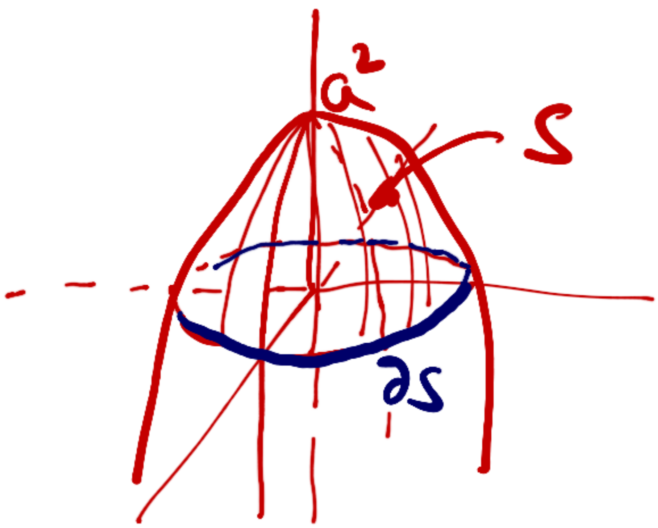
$$\underbrace{\oint_{\partial S} F \cdot dr}_{\text{line integral for a vector field.}} = \underbrace{\int_S \operatorname{curl} F \cdot dS}_{\text{surface integral.}}$$

It relates what happens on the border of the surface to what happens through the surface.

Example: Problem 7 i). $\boxed{4.4}$

Compute $\displaystyle\int_S \operatorname{curl} F \cdot dS$ for $F(x,y,z) = (x^2y^2, yz, xy)$

$$S = \{ z = a^2 - x^2 - y^2,\ z \geq 0 \}$$



We might compute the curl F and parametrise the surface and then obtain

$$\int_S \operatorname{curl} F \cdot dS = \oint_{\partial S} F \cdot dr.$$

Stokes' Th

Border of S

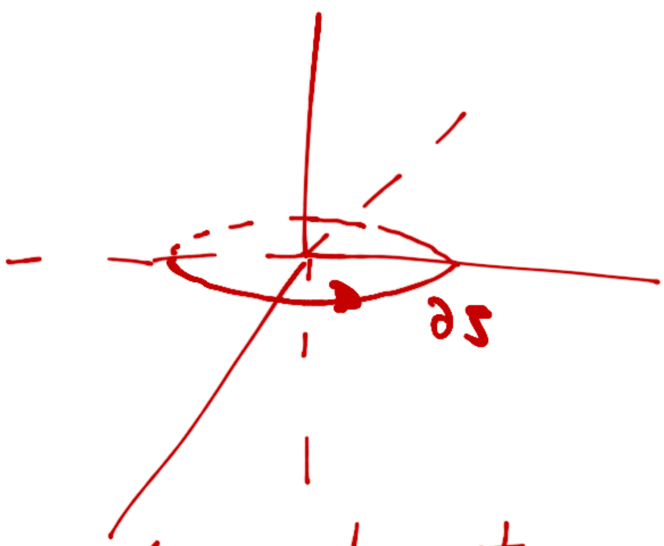
$$\partial S = \{ (x,y,z) \in \mathbb{R}^3,\ x^2+y^2=a^2,\ z=0 \}$$

Parametrise ∂S

$$\sigma(t) = (a\cos t, a\sin t, 0)$$

$$\sigma'(t) = (-a\sin t, a\cos t, 0)$$

$t \in [0, 2\pi]$ with positive orientation.



$$F(\sigma(t)) = (a^4\cos^2 t\sin^2 t, 0, a^2\cos t\sin t)$$

$$\oint_{\partial S} F \cdot d\mathbf{r} = \int_0^{2\pi} \underbrace{F(\sigma(t)) \cdot \sigma'(t)} \, dt$$

$$= \int_0^{2\pi} \underbrace{\left(a^4 \cos^2 t \sin^2 t, \; 0, \; a^2 \sin t \cos t\right)}_{F(\sigma(t))} \cdot \overbrace{\left(-a \sin t, a \cos t, 0\right)}^{\sigma'(t)} \, dt$$

$$= -a^5 \int_0^{2\pi} \cos^2 t \, \underline{\sin^3 t} \, dt =$$

$$= -a^5 \int_0^{2\pi} \cos^2 t \left(1 - \cos^2 t\right) \sin t \, dt$$

$$= -a^5 \int_0^{2\pi} \left(\underline{\cos}^2 t \sin t - \underline{\cos}^4 t \sin t\right) dt$$

$$= -a^5 \left[\left(-\frac{\cos^3 t}{3} + \frac{\cos^5 t}{5} \right) \right]_0^{2\pi} = 0$$

Then

$$\int_S \operatorname{curl} F \cdot ds = 0 = \oint_{\partial S} F \cdot dr.$$

$$-\frac{\cos^3 2\pi}{3} + \frac{\cos^3 0}{3}$$

$$\frac{\cos^5 2\pi}{5} - \frac{\cos^5 0}{5}$$

Gauss' Theorem (Divergence Th.)

Extension to $\mathbb{R}^N$ of the Fundamental Theorem of Calculus and Green's Theorem.

Assuming closed surface.

Let Ω be an open, bounded domain in $\mathbb{R}^3$ and $\partial\Omega$ is a closed surface.

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ vector field of class C^1

$\partial\Omega$ with positive orientation (outward normal vector n)

Then,

$$\underbrace{\int_{\partial\Omega} F\cdot n \, dS}_{\substack{\text{surface}\\ \text{integral for}\\ \text{scalar field}}} = \underbrace{\int_{\Omega} \operatorname{div} F \, dV}_{\text{triple integral.}}$$

where $F = (F_1, F_2, F_3)$ and

$$\boxed{\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

- Gauss' Th relates how a vector field goes through a surface, with what happens in the interior of the domain Ω, for a scalar function $\operatorname{div} F$

Interpretation of divergence

$$\int_{\partial\Omega} \underbrace{F\cdot n}_{\text{orthogonal direction.}}\, dS \overset{\text{Gauss' TR.}}{=} \int_{\Omega} \operatorname{div} F\, dV = \operatorname{div} F(x_0 y_0 z_0)\cdot \operatorname{Vol}\Omega$$

mean value Theorem for integrals.

$(x_0 y_0 z_0) \in \Omega$

Then,

$$\operatorname{div} F(x_0, y_0 z_0) = \frac{1}{\operatorname{Vol}\Omega} \int_{\partial\Omega} F\cdot n\, dS.$$

divergence ≡ ratio of the flow F going through the boundary $\partial\Omega$ per unit of volume

In application terms

$\operatorname{div} F \equiv$ represents the rate of expansion per unit volume.

Example: Problem 17

$$\Omega = \{ x^2+y^2+z^2 \leq a^2 \}$$

$$S = \partial\Omega = \{ x^2+y^2+z^2 = a^2 \}$$

$$F(x,y,z) = \Big(\underbrace{\sin(yz)+e^{z}}_{F_1}, \underbrace{x\cos z + \log(1+x^2+z^2)}_{F_2}, \underbrace{e^{x^2+y^2+z^2}}_{F_3} \Big)$$

$$\int_S F\cdot dS \;?$$

Applying Gauss' Theorem

$$\int_S F \cdot dS = \int_\Omega \operatorname{div} F \, dV = \int_\Omega \operatorname{div} F \, dx\,dy\,dz$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 0 + 0 + 2z\,e^{x^2+y^2+z^2}$$

Then,

$$\int_\Omega \operatorname{div} F \, dV = \int_\Omega 2z\,e^{x^2+y^2+z^2} \, dx\,dy\,dz$$

Using spherical coordinates:

$$\left.\begin{aligned} x &= r\cos\theta\sin\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\varphi \end{aligned}\right| \quad \begin{aligned} &|JT| = r^2\sin\varphi \\ &0 \le \theta \le 2\pi \\ &0 \le \varphi \le \pi \\ &0 \le r \le a \end{aligned}$$

$$\int_\Omega \operatorname{div} F \, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 2\underbrace{r\cos\varphi}_{z}\ \underbrace{e^{r^2}}_{e^{x^2+y^2+z^2}} \cdot \underbrace{r^2\sin\varphi}_{|JT|} \, dr\, d\varphi\, d\theta$$

$$= 4\pi \underbrace{\left(\int_0^a r^3 e^{r^2}\, dr\right)}_{\text{complicated.}} \underbrace{\left(\int_0^{\pi} \cos\varphi\sin\varphi\, d\varphi\right)}_{\left.\frac{\sin^2\varphi}{2}\right]_0^{\pi} = 0}$$

$$= 0$$

$$\int z^2 \cdot \underbrace{z e^{z^2}} \, dz = \frac{z^2 e^{z^2}}{2} - \underbrace{\int z e^{z^2} \, dz}_{= \frac{1}{2} e^{z^2}}$$

$$u = z^2 \Rightarrow du = 2z$$

$$dv = \underline{z\, e^{z^2}} \Rightarrow v = \frac{1}{\underline{2}} e^{z^2}$$

Parametrize $S = \partial \Omega = \{ x^2 + y^2 + z^2 = a^2 \}$

$$\left. \begin{aligned} x &= a \cos\theta \sin\varphi \\ y &= a \sin\theta \sin\varphi \\ z &= a \cos\varphi \end{aligned} \right\} : \Phi$$

$$\Phi(\theta, \varphi) = \left(a\cos\theta \sin\varphi, \underline{a \sin\theta \sin\varphi}, \underline{a \cos\varphi} \right)$$

$$\int_S F\, dS = \underbrace{\int_0^{2\pi}\int_0^{\pi}}_{\text{parametric region.}} F\big(\Phi(\theta,\varphi)\big)\cdot\big(T_\theta \times T_\varphi\big)\, d\varphi\, d\theta$$

$$F(x,y,z) = \left(\sin yz + e^{z},\ x\cos z + \log(1+x^2+z^2),\ e^{x^2+y^2+z^2}\right)$$

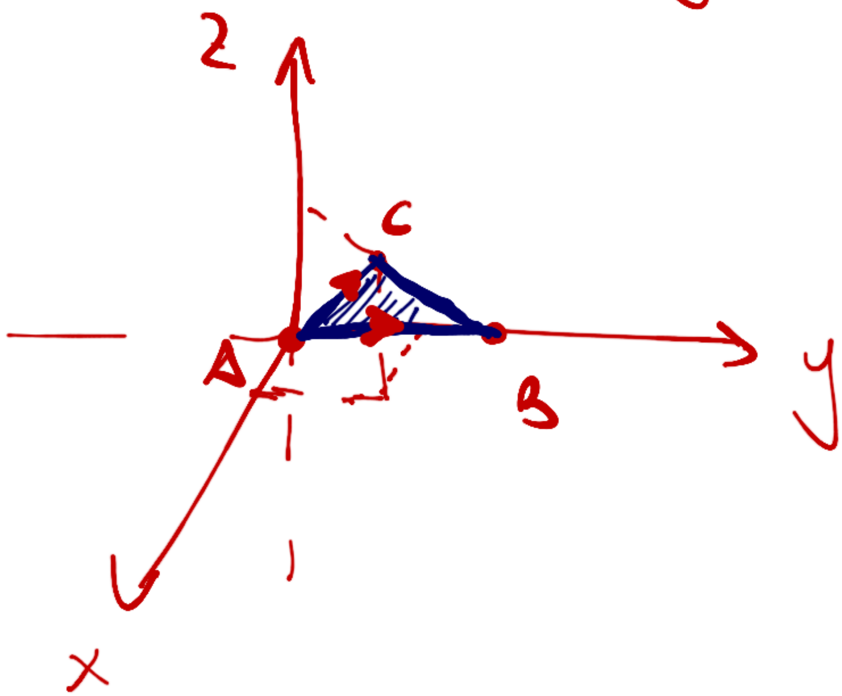
$$F\big(\Phi(\theta,\varphi)\big) = \left(\boxed{\sin\left(a^2 \sin\theta \sin\varphi \cos\varphi\right) + e^{a\cos\varphi}} \ldots\right.$$

Problem 9 - 4.4

$$F(x,y,z) = (2y, 3z, x)$$

Triangle : $\underbrace{A(0,0,0),\ B(0,2,0)\ C(1,1,1)}_{T}$

i) Choose an orientation for the surface. T

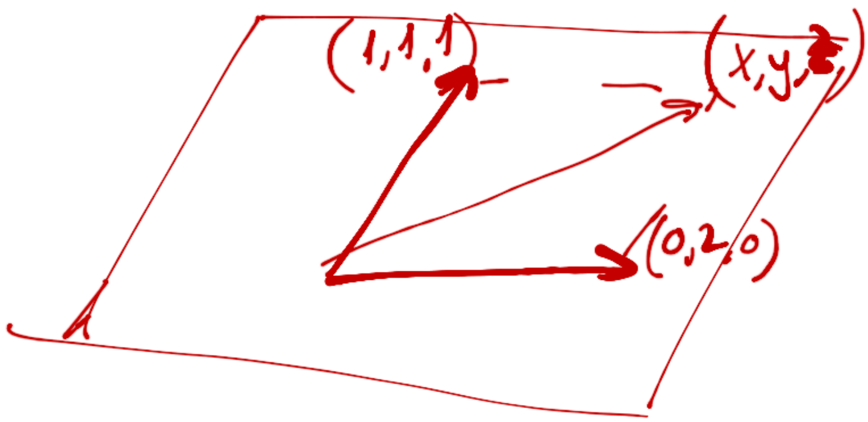


First we write the equation of the plane.

Plane:

$$\begin{vmatrix} x & y & z \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow 2x - 2z = 0$$

$$\boxed{x - z = 0}$$



$$(x\,y\,z) = \lambda(1,1,1) + \mu(0,2,0)$$

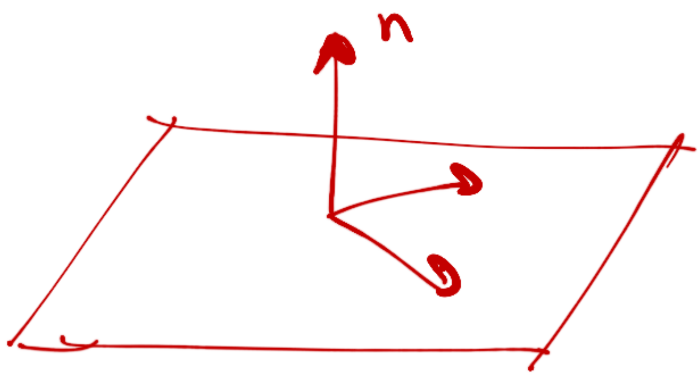
$$\lambda(1,1,1) + \mu(0,2,0) - (x,y,z) = 0$$

$$\lambda, \mu \neq 0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

$n \cdot (x, y, z) = 0$ if $n \perp (x, y, z)$

$n \equiv$ normal vector to the plane.



If $x - z = 0$

$$n = (1, 0, -1), \quad \|n\| = \sqrt{2}$$

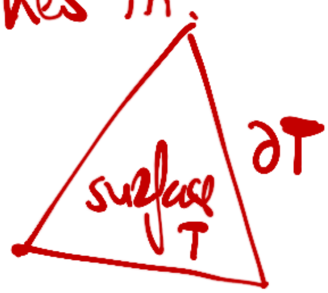
Unitary normal vector

$$\tilde{n} = \frac{1}{\sqrt{2}}(1, 0, -1)$$

unitary normal vector for the triangle ΔABC

ii) Compute the path integral of F along ∂T

$$\oint_{\partial T} F \cdot dr \underset{\text{stokes' TR}}{=} \int_T \text{curl} F \cdot ds$$



Surface T with a close boundary ∂T

$$\text{curl} F = \text{rot} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & x \end{vmatrix} =$$

$$= i(0-3) - j(1-0) + k(-2) = (-3, -1, -2)$$

$$\int_T \mathrm{rot}F \, ds = \int_T \mathrm{rot}F \cdot n \, ds$$

$$= \int_T (-3,-1,-2) \cdot \frac{1}{\sqrt{2}}(1,0,-1) \, ds$$

$$= \underbrace{\int_T \frac{-3+2}{\sqrt{2}} \, ds}_{-\frac{1}{\sqrt{2}} \int_T ds} = \underbrace{\frac{-1}{\sqrt{2}} \cdot \text{area of a Triangle}}_{\boxed{-1}}$$

$$\text{Area of } T = \frac{1}{2} \| AB \times AC \| = \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} \right\|$$

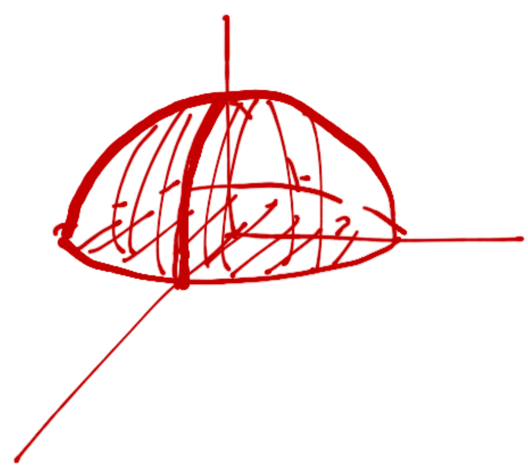
$$= \frac{1}{2} \|(2,0,-2)\| = \frac{\sqrt{8}}{2} = \sqrt{2}$$

Problem 10

$$F(x,y,z) = \Big(\overbrace{y\sin(x^2+y^2)}^{F_1}, \overbrace{-x\sin(x^2+y^2)}^{F_2}, \overbrace{z(3-2y)}^{F_3}\Big)$$

Region $W = \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2 \le 1,\ z \geq 0\}$

$$\oint_{\partial W} F \quad ?$$



$$\Vert$$

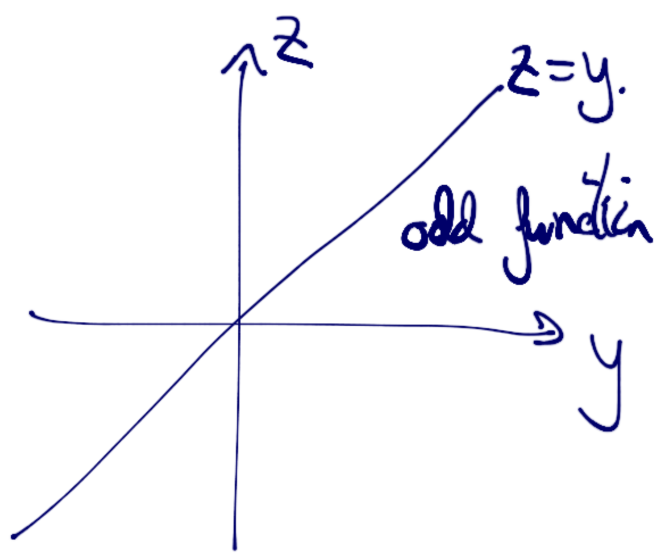
$$\int_W \operatorname{div} F \, dV$$ by Gauss' Th.

because ∂W is a closed surface.

$$\operatorname{div} F = \overbrace{\cancel{2xy\cos(x^2+y^2)}}^{\partial F_1/\partial x} + \underbrace{\cancel{(-2xy)\cos(x^2+y^2)}}_{\partial F_2/\partial y} + \overbrace{(3-2y)}^{\partial F_3/\partial z}$$

$$\int_W \operatorname{div} F \, dx\,dy\,dz = \int_W (3-2y)\, dx\,dy\,dz$$

$$= 3\int_W dx\,dy\,dz - 2\underbrace{\int_W y\, dx\,dy\,dz}_{\text{By symmetry it is zero.}}$$



$$= 3 \underbrace{\text{Volume } W}_{\frac{1}{2}\cdot\frac{4\pi 2^3}{3}}$$

$2=1$

$$= 3\,\frac{1}{2}\,\frac{4\pi}{3} = \boxed{2\pi}$$

and W is symmetric with respect to z

Problem 11 - 4.4

Verify Stokes' Th.

$$\int_S \text{rot} F \cdot n \, dS = \oint_{\partial S} F \cdot dr.$$

For the line integral:

i) $F(x,y,z) = (y^2, xy, xz)$

$S = \{ \text{paraboloid } z = a^2 - x^2 - y^2, \ z \geq 0 \}$

$\partial S = \{ x^2 + y^2 = a^2, \ z = 0 \}$

Parametrization of ∂S.

$\sigma(t) = (a\cos t, a\sin t, 0) \quad t \in [0, 2\pi]$

$$\boxed{\oint_{\partial S} F\, dr = \int_0^{2\pi} F(\sigma(t))\cdot \sigma'(t)\, dt}$$

For the surface:

We use the Monge parametrization:

$$z = a^2 - x^2 - y^2$$

$$\left.\begin{aligned} x &= u \\ y &= v \\ z &= a^2 - u^2 - y^2 \end{aligned}\right\} \quad (u,v) \in D$$

$$D = \{ u^2 + v^2 \leq a^2 \}$$ Parametric region.

$$T_u \times T_v = \left(-\frac{\partial z}{\partial u}, -\frac{\partial z}{\partial v}, 1 \right) = (2u, 2v, 1)$$

$$\int_S \operatorname{rot} F \, dS = \iint_D \operatorname{rot} F\big(\Phi(u,v)\big) \cdot \big(T_u \times T_v\big)\, du\, dv$$